Bethe-ansatz solution of the t-J model

1990 J. Phys. A: Math. Gen. 23 L409

(http://iopscience.iop.org/0305-4470/23/9/002)

J. Phys. A: Math. Gen. 23 (1990) L409–L414. Printed in the UK

LETTER TO THE EDITOR

Bethe-ansatz solution of the t-J model

Sarben Sarkar

Centre for Theoretical Studies, Royal Signals and Radar Establishment, Great Malvern, Worcs WR14 3PS, UK

Received 19 January 1990

Abstract. The t-J model for strongly correlated electron systems is mapped onto a model with a generalised permutation operator. For a specific value of the ratio t/J this model in one space dimension is shown to be equivalent to that of Lai and can be solved by the Bethe ansatz. This special point corresponds to supersymmetric invariance of the Hamiltonian.

There is currently intense interest in the phenomenon of high-temperature superconductivity and several theoretical models have been proposed. One is based on the belief that the repulsion energy U for two electrons located on the same atom is much larger than the bandwidth ($8t$) of the electrons. A natural consequence of this is that, for an occupation of one electron per site, the electrons are essentially localised. The energetics of the spin degrees of freedom $\boldsymbol{S}$ are determined by a nearest-neighbour Heisenberg interaction which favours some sort of antiferromagnetism. For La_2CuO_4 and $YBa_2Cu_3O_6$ antiferromagnetic long-range order is indeed observed, although the introduction of sufficient doping eventually forces the Néel temperature to zero. A simple and very popular model (Hirsch 1985, Anderson 1988) which describes both the Heisenberg interaction and hopping in the presence of holes is the t-J model with Hamiltonian H given by

$$H = t \sum_{\langle i,i' \rangle} (X_{i'}^{\sigma 0} X_i^{0\sigma} + X_i^{\sigma 0} X_{i'}^{0\sigma}) + J \sum_{\langle i,i' \rangle} (\boldsymbol{S}_i \cdot \boldsymbol{S}_{i'} - \tfrac{1}{4}) \delta_{n_i 1} \delta_{n_{i'} 1} . \tag{1}$$

Moreover H acts in the Hilbert space of states where no double occupancy of sites is allowed. Here J ($=2t^2/U$) is the exchange interaction and $\langle \boldsymbol{i}, \boldsymbol{i}' \rangle$ denotes that $\boldsymbol{i}$ and $\boldsymbol{i}'$ are nearest-neighbour sites on a lattice whose dimensionality is as yet unspecified. The operator $X_i^{0\sigma}$ destroys an electron with spin $S_z = \sigma$ and leaves a hole at site $\boldsymbol{i}$. The Heisenberg exchange interaction is only operative if the sites $\boldsymbol{i}$ and $\boldsymbol{i}'$ are occupied and this explains the presence of Kronecker δ-functions in (1) (n_i being the number of carriers at site $\boldsymbol{i}$). Since electrons have spin-$\frac{1}{2}$ it is convenient to rewrite the exchange interaction in terms of Pauli spin matrices $\boldsymbol{\sigma}$ to give

$$J(\boldsymbol{S}_i \cdot \boldsymbol{S}_{i'} - \tfrac{1}{4}) = \tfrac{1}{4} J(\boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_{i'} - 1). \tag{2}$$

Implicit in our discussion has been the assumption that the t-J model is related to the celebrated Hubbard model which also describes the effect on hopping of correlation energy. Indeed from degenerate perturbation theory in t/U (Pike *et al* 1990) the t-J model, augmented by three site-assisted hopping terms, can be derived from the Hubbard model. J is necessarily then much smaller than t. However, we are interested in cases where $J \neq 2t^2/U$ and in particular $J \sim t$. From this point of view it is important to note that J is somewhat larger than $2t^2/U$ when the t-J model is derived from a multiband Hubbard model (Zhang and Rice 1988). The Hubbard model was solved in one space dimension by Lieb and Wu (1968). It is thus natural to examine whether the t-J model in one dimension can be solved by a similar technique. The solution does not follow from that of the Hubbard model both because of the assisted hopping terms and because J is not necessarily much smaller than t. We find that it is indeed possible to solve the t-J model by the Bethe ansatz. However, the solution is not so straightforward and goes through only for a specific choice of the ratio t/J.

In this letter we will give the ideas behind the Bethe ansatz solution leaving the details and physical consequences of the solution to a later publication. The operators $X^{0\alpha}$ allow us to extend the SU(2) algebra of spins to the superalgebra U(1/2) (Wiegmann 1988, Cornwell 1989, Sarkar 1990). It will be helpful to introduce a harmonic oscillator representation of the superalgebra (Bars and Günaydin 1983) with

$$X^{0\alpha} = f^{\alpha} b^{\dagger} \tag{3}$$

b being a bosonic and f^{α} a fermionic annihilation operator. The t term can then be written as

$$\sum_i t(X_i^{\sigma 0} X_{i+1}^{0\sigma} + X_{i+1}^{\sigma 0} X_i^{0\sigma}) = \sum_i t(b_i f_i^{\sigma\dagger} f_{i+1}^{\sigma} b_{i+1}^{\dagger} + b_{i+1} f_{i+1}^{\sigma\dagger} f_i^{\sigma} b_i^{\dagger}). \tag{4}$$

Clearly f_i^{σ} destroys a spin σ at site i and now we have an explicit operator $b_i^{\dagger}$ which creates a state of no occupation at site i. The exchange part of H we will rewrite as

$$\frac{J}{2}\sum_i (P_{i,i+1} - 1) \qquad \text{where} \qquad P_{i,i+1} = \tfrac{1}{2}(\boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_{i+1} + 1) \tag{5}$$

and the summation is effective only over sites i, $i+1$ which both have spins occupying them. It is possible to interpret (4) and $P_{i,i+1}$ as permutation operators. This is crucial and only possible owing to (3) which permits us to consider each site on the lattice as being occupied by a single particle, whether it be a boson or a spin-up ($\uparrow$) fermion or spin-down ($\downarrow$) fermion. The permutation operator nature of H can be easily established by considering three simple examples. Let us first consider a two-site lattice with one hole and one up-spin, a general state vector for which can be written as

$$|\psi\rangle = \alpha b_1^{\dagger} f_2^{\uparrow\dagger}|0\rangle + \beta b_2^{\dagger} f_1^{\uparrow\dagger}|0\rangle. \tag{6}$$

$|0\rangle$ is a vacuum but is *not* equivalent to a two-hole state which is given by $b_1^{\dagger} b_2^{\dagger}|0\rangle$.

Now it is easy to calculate that

$$H|\psi\rangle = t(\alpha b_2^{\dagger} f_1^{\uparrow\dagger}|0\rangle + \beta b_1^{\dagger} f_2^{\uparrow\dagger}|0\rangle) \tag{7}$$

and so we see that

$$t(b_i b_{i+1}^{\dagger} f_i^{\sigma\dagger} f_{i+1}^{\sigma} + b_{i+1} b_i^{\dagger} f_{i+1}^{\sigma\dagger} f_i^{\sigma}) = t(P_{i,i+1}^{(0,\uparrow)} + P_{i,i+1}^{(0,\downarrow)}) \tag{8}$$

where

$$P_{i,i+1}^{(0,\sigma)}(\ldots f_i^{\sigma\dagger} b_{i+1}^{\dagger})|0\rangle = \delta_{\sigma\gamma}(\ldots f_{i+1}^{\sigma\dagger} b_i^{\dagger})|0\rangle \tag{9}$$

for a lattice with any number of sites.

For the second example we consider a two-site lattice with one up-spin and one down-spin. The general state can be written as

$$|\psi\rangle = \alpha f_1^{\uparrow\dagger} f_2^{\downarrow\dagger}|0\rangle + \beta f_1^{\downarrow\dagger} f_2^{\uparrow\dagger}|0\rangle \tag{10}$$

and then

$$H|\psi\rangle = \tfrac{1}{2}J(\alpha f_1^{\downarrow+}f_2^{\uparrow+}|0\rangle + \beta f_1^{\uparrow+}f_2^{\downarrow+}|0\rangle - \alpha f_1^{\uparrow+}f_2^{\downarrow+}|0\rangle - \beta f_1^{\downarrow+}f_2^{\uparrow+}|0\rangle). \tag{11}$$

Similarly, for a two-site lattice with both spins in the same direction $S_z = \sigma$, the general state is

$$|\psi\rangle = f_1^{\sigma+}f_2^{\sigma+}|0\rangle \tag{12}$$

and

$$H|\psi\rangle = 0. \tag{13}$$

Consequently we have for any size of lattice

$$P_{i,i+1}(\ldots f_i^{\alpha+}f_{i+1}^{\beta+}|0\rangle) = \ldots f_i^{\beta+}f_{i+1}^{\alpha+}|0\rangle. \tag{14}$$

These arguments clearly hold for lattices of higher dimension.

For the case of the Heisenberg antiferromagnet the fact that the Hamiltonian could be written as a permutation operator was noticed by Bethe (1931). For the *t-J* model we have now shown a generalisation of this.

Let us examine under what conditions the model is soluble using the Bethe ansatz. For a one-dimensional lattice with N sites and occupied by $(N-2)$ up-spins, one down-spin and one hole, we can write the state ket as

$$|\psi\rangle = \sum_{x_1,x_2} \alpha(x_1,x_2) f_1^{\uparrow+}f_2^{\uparrow+}\ldots f_{x_1-1}^{\uparrow+}f_{x_1}^{\downarrow+}f_{x_1+1}^{\uparrow+}\ldots b_{x_2}^{+}\ldots f_N^{\uparrow+}|0\rangle \tag{15}$$

x_1 being the position of the down-spin and x_2 the position of the hole. The Schrödinger equation is

$$H|\psi\rangle = E|\psi\rangle. \tag{16}$$

The situation is simplest when x_1 and x_2 are far apart. The coefficient of the term

$$f_1^{\uparrow+}f_2^{\uparrow+}\ldots f_{x_1-1}^{\uparrow+}f_{x_1}^{\downarrow+}f_{x_1+1}^{\uparrow+}\ldots b_{x_2}^{+}\ldots f_N^{\uparrow+}|0\rangle$$

in (16) gives

$$\begin{aligned} E\alpha(x_1,x_2) = {}& -J\alpha(x_1,x_2) + \tfrac{1}{2}J[\alpha(x_1+1,x_2) + \alpha(x_1-1,x_2)] \\ & + t(\alpha(x_1,x_2-1) + \alpha(x_1,x_2+1)). \end{aligned} \tag{17}$$

Since our purpose is to solve the model with the Bethe ansatz it would be natural to try the standard prescription (Andrei *et al* 1983) and write

$$\alpha(x_1,x_2) = \sum_{\substack{P,Q\\ \epsilon S_2}} A_P(Q)\exp\left(\mathrm{i}\sum_{j=i}^{2} k_{Pj}x_{Qj}\right)\theta(x_Q). \tag{18}$$

Here P and Q are permutations belonging to S_2 the permutation group over two objects, k_j are 'quasimomenta' to be determined, $A_P(Q)$ are some coefficients, and $\theta(x_Q)$ defines the sector

$$x_{Q1} < x_{Q2}. \tag{19}$$

Equation (17) implies that

$$\begin{aligned} E(A_1(1)&\exp[\mathrm{i}(k_1x_1+k_2x_2)] + A_2(1)\exp[\mathrm{i}(k_2x_1+k_1x_2)] \\ &= 2(\tfrac{1}{2}J\cos k_1 + t\cos k_2)A_1(1)\exp[\mathrm{i}(k_1x_1+k_2x_2)] \\ &\quad + 2(\tfrac{1}{2}J\cos k_2 + t\cos k_1)A_2(1)\exp[\mathrm{i}(k_2x_1+k_1x_2)] \end{aligned} \tag{20}$$

where $Q=1$ is a shorthand for $Q = \binom{12}{12}$ and $Q=2$ denotes $Q=\binom{12}{21}$.

A similar convention is adopted for P. Equation (20) is not consistent unless

$$\frac{J}{2t}=1.$$

The sign of t in H can be changed at will by making the canonical transformation

$$b_i \to (-1)^i b_i$$

and so effectively we have the condition

$$\left|\frac{J}{2t}\right| = 1. \tag{21}$$

For the Bethe ansatz method the next stage is to consider when x_1 and x_2 are nearest-neighbour sites. The Schrödinger equation then implies that

$$\begin{aligned} A_1(2) &= u^{12}A_2(2)+v^{12}A_2(1) \\ A_1(1) &= u^{12}A_2(1)+v^{12}A_2(2) \end{aligned} \tag{22}$$

where u_{12} and v_{12} are functions $u(k_1, k_2)$ and $v(k_1, k_2)$ of k_1 and k_2. The difficulty of obtaining a solution via the Bethe ansatz is in satisfying consistency conditions for these coefficients (Andrei *et al* 1983). In particular we need

$$u(k_1, k_2)v(k_2, k_1)+v(k_1, k_2)u(k_2, k_1)=0. \tag{23}$$

A detailed calculation shows that this consistency condition fails to hold.

In this particular problem we will show that the situation can be retrieved by being less straightforward about the application of the Bethe ansatz. Lai (1974) considered a problem involving a permutation operator. His permutation operators P_L, when expressed in our notation, satisfy

$$P_{Li,i+1} f_i^{\alpha\dagger} f_{i+1}^{\alpha\dagger}|0\rangle = f_{i+1}^{\alpha\dagger} f_i^{\alpha\dagger}|0\rangle \tag{24}$$

and

$$P_{Li,i+1} f_i^{\uparrow\dagger} f_{i+1}^{\downarrow\dagger}|0\rangle = f_i^{\downarrow\dagger} f_{i+1}^{\uparrow\dagger}|0\rangle. \tag{25}$$

Clearly (24) differs from (14) by a minus sign. In order to emphasise this difference we define in the obvious fashion $P^{(\uparrow\uparrow)}_{Li,i+1}$, $P^{(\downarrow\downarrow)}_{Li,i+1}$ and $P^{(\downarrow\uparrow)}_{Li,i+1}$, and also the corresponding Ps for our definition of the permutation operator. We then have

$$\begin{aligned} P_{i,i+1} &= P^{(\uparrow\uparrow)}_{i,i+1}+P^{(\downarrow\downarrow)}_{i,i+1}+P^{(\downarrow\uparrow)}_{i,i+1} \\ &= -(P^{(\uparrow\uparrow)}_{Li,i+1}+P^{(\downarrow\downarrow)}_{Li,i+1}-P^{(\downarrow\uparrow)}_{Li,i+1}). \end{aligned} \tag{26}$$

Lai has shown that the Hamiltonian H_L

$$H_L = -\sum_{i,\gamma} P^{(0,\gamma)}_{i,i+1} - \sum_i (P^{(\uparrow\uparrow)}_{Li,i+1}+P^{(\downarrow\downarrow)}_{Li,i+1}+P^{(\downarrow\uparrow)}_{Li,i+1}) - \sum_i P^{(0,0)}_{Li,i+1} \tag{27}$$

can be solved using the Bethe ansatz. The H that we derived from the t-J model in this notation is

$$H = t\sum_{i,\gamma} P^{(0,\gamma)}_{i,i+1} - \frac{J}{2}\sum_i (P^{(\uparrow\uparrow)}_{Li,i+1}+P^{(\downarrow\downarrow)}_{Li,i+1}+P^{(0,0)}_{Li,i+1}-P^{(\downarrow\uparrow)}_{Li,i+1}) \tag{28}$$

up to an unimportant constant. (The operator $P^{(0,0)}_{Li,i+1}$ satisfies $P^{(0,0)}_{Li,i+1} b_i^\dagger b_{i+1}^\dagger|0\rangle = b_{i+1}^\dagger b_i^\dagger|0\rangle$.)

The reason for the difference between P_L and P is the fermionic nature of f^σ. In one dimension the difference between fermions and bosons is somewhat artificial and in fact it is possible to make a canonical Jordan–Wigner transformation to go from one to another. The result of such a transformation on H is

$$H \to H' = t \sum_{i,\gamma} P_{i,i+1}^{(0,\gamma)} + \frac{J}{2} \sum_i (P_{Li,i+1}^{(\uparrow\uparrow)} + P_{Li,i+1}^{(\downarrow\downarrow)} + P_{Li,i+1}^{(\downarrow\uparrow)} + P_{Li,i+1}^{(0,0)}). \tag{29}$$

The eigenvalues of H' and H are the same since the transformation is canonical. For $t = J/2 = 1$ the similarity with (27) is striking but there is a difference of an overall minus sign and of course the hole operator is a fermion and the spin operators are bosons.

Fortunately there is a simple isomorphism between the eigenstates of H' and H_L. A product with any totally antisymmetric function of the coordinates of the particles on the lattice will convert the eigenstates of H_L into those of H'. Clearly the energy eigenvalues are the same.

For the situation described by (15) the Schrödinger equation for H_L gives the equations

$$\begin{aligned}[2(\cos k_1 + \cos k_2) + 1]&\alpha(x_1, x_1+1) \\ &= \alpha(x_1+1, x_1) + \alpha(x_1, x_1+2) + \alpha(x_1-1, x_1+1)\end{aligned} \tag{30}$$

and

$$\begin{aligned}[2(\cos k_1 + \cos k_2) + 1]&\alpha(x_1+1, x_1) \\ &= \alpha(x_1, x_1+1) + \alpha(x_1+1, x_1-1) + \alpha(x_1+2, x_1).\end{aligned} \tag{31}$$

In (22)

$$u_{12} = \frac{(1+e^{ik_1})(1+e^{ik_2})}{1+2\,e^{ik_2}+e^{i(k_1+k_2)}} \qquad \text{and} \qquad v_{12} = \frac{e^{ik_1} - e^{ik_2}}{1+2\,e^{ik_2}+e^{i(k_1+k_2)}} \tag{32}$$

(23) and other conditions necessary for the Bethe ansatz are satisfied. For more complicated situations than (15), e.g. with down-spins at $x_1, \ldots, x_{n_\downarrow}$ and holes at $x_{n_\downarrow+1}, \ldots, x_{n_\downarrow+n_0}$ the Bethe ansatz requires

$$\alpha(x_1, \ldots, x_{n_\downarrow+n_0}) = \sum_{\substack{P,Q \\ \in S_{n_\downarrow+n_0}}} A_P(Q) \exp\left[i \sum_{j=1}^{n_0+n_\downarrow} (k_{P_j} x_{Q_j}) \right] \theta(x_Q) \tag{33}$$

(where $\theta(x_Q)$ requires $x_{Q1} < x_{Q2} < \ldots < x_{Q(n_0+n_\downarrow)}$).

We will conclude by examining the permutation symmetric properties of the ground state. Lai and Yang (1971) discussed a mixture of interacting spin-$\frac{1}{2}$ fermions and bosons moving in a one-dimensional continuum. Despite the considerable differences in detail between their model and the one discussed here, their argument for the permutation symmetry of $\alpha(x_1, \ldots, x_{n_\downarrow+n_0})$ is independent of such details. The difference between our problem and that of Lai and Yang manifests itself only in their form of u_{12} and v_{12} and in the fact that they have k also associated with the up-spins. Hence $\alpha(x_1, \ldots, x_{n_\downarrow+n_0})$ transforms as the irreducible representation of S_N found by Lai and Yang. The representation may be labelled by the Young tableau

$$(2+n_0, 2^{n_\downarrow-1}, 1^{N-2n_\downarrow-n_0}).$$

We have for definiteness assumed that $N - 2n_\downarrow - n_0 \geqslant 0$.

We may wonder as to why we could only solve the t-J model for a special ratio of t/J. Is there a special symmetry? The answer is in the affirmative and the symmetry is the supersymmetry that we discussed earlier. A general form of H can be written as

$$H = g \sum_i (X_i^{\sigma 0} X_{i+1}^{0\sigma} + X_{i+1}^{\sigma 0} X_i^{0\sigma}) + g' \sum_i X_i^{\alpha\beta} X_{i+1}^{\beta\alpha} + g'' \sum_i X_i^{00} X_{i+1}^{00}. \tag{34}$$

(The t-J model corresponds to $g = t$, $g' = J/2$ and $g'' = -J/2$).

$X_i^{\alpha\beta}$ changes the spin β at site i to spin α and X_i^{00} counts whether there is a hole at site i. A necessary and sufficient condition for global supersymmetry is

$$\left(\sum_j X_j^{0\alpha}, H\right) = 0. \tag{35}$$

Equation (35) implies that

$$g = -g' = g''$$

which is the condition necessary for our Bethe ansatz solution.

The Bethe ansatz equations for the energy eigenvalues can be obtained by imposing periodic boundary conditions on the wavefunction. They may be solved in the usual way by the so-called generalised Bethe hypothesis (Yang 1967, Sutherland 1968), details of which will be presented elsewhere.

I would like to thank A C Hewson and N Andrei for illuminating discussions on the Bethe ansatz, J H Jefferson for a critical reading of the manuscript and NATO for a collaborative research grant.

References

Anderson P W 1988 *Frontiers and Borderlines in Many Particle Physics* ed R A Broglia and J R Schrieffer (Amsterdam: North-Holland)
Andrei N, Furuya K and Lowenstein J H 1983 *Rev. Mod. Phys.* **55** 331
Bars I and Günaydin M 1983 *Commun. Math. Phys.* **91** 31
Bethe H 1931 *Z. Phys.* 71 205
Cornwell J F 1989 Supersymmetries and Infinite Dimensional Algebras *Group Theory in Physics* vol 3 (New York: Academic)
Hirsch J E 1985 *Phys. Rev. Lett.* **54** 1317
Lai C K 1974 *J. Math. Phys.* **15** 1675
Lai C K and Yang C N 1971 *Phys. Rev.* A **3** 393
Lieb E H and Wu F Y 1968 *Phys. Rev. Lett.* **20** 1445
Pike E R, Jefferson J H and Sarkar S 1990 *An Introduction to Electron Correlations and High Temperature Superconductivity* (Bristol: Hilger) to be published
Sarkar S 1990 Supercoherent States for the *t-J* Model *Preprint* RSRE
Sutherland B 1968 *Phys. Rev. Lett.* **20** 98
Wiegmann P B 1988 *Phys. Rev. Lett.* **60** 821
Yang C N 1967 *Phys. Rev. Lett.* **19** 1312
Zhang F C and Rice T M 1988 *Phys. Rev.* B **37** 3759